

The fractal dimension of the Riemann zeta zeros

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ABSTRACT: In this paper, we consider the nontrivial zeros of the Riemann zeta function as the eigenvalues of the Dirac operator on a fractal manifold. From the heat kernel expansion, we figure out that the fractal dimension of the manifold is about 1.1-1.2. Also we compare this result to the random matrix theory and the quantum chaos theory.

KEYWORDS: Riemann hypothesis, the fractal dimension, random matrix theory, quantum chaos.

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1. Introduction

The Riemann hypothesis(RH)[1] is that all the nontrivial zeros of the zeta function have real part $1/2$, so that the quantities t_j defined by

$$\zeta(\frac{1}{2} + it_j) = 0 \tag{1.1}$$

are all real. There are strong evidence to support this hypothesis: the first billions of zeros are on the line, and also more than 40% of the nontrivial zeros satisfy the Riemann hypothesis. There are also mathematical evidence, such as the Deligne's proof of the corresponding Riemann hypothesis for Zeta function of arbitrary varieties over finite fields, or the Weil's explicit formula. For a review, see the paper written for the Millennium by E.Bombieri[2], or the paper[3] by J.B.Conrey.

Polya and Hilbert made the conjecture that the imaginary part of the Riemann zeros could be the eigenvalues of a Hermitian operator. Then the RH would follow, since Hermitian operator have real eigenvalues. In the 50's Selberg found a remarkable duality between the length of the geodesics on a Riemann surface and the eigenvalues of the Laplacian operator defined on it[4]. Quite independently of Selberg's work, Montgomery showed that the Riemann zeros are distributed randomly

and obeying locally the statistical law of the Random Matrix Theory(RMT)[5]. The matrix related to the Riemann zeros is the gaussian unitary ensemble(GUE) associated to the random hermitian matrices. Montgomery's results found an impressive numerical confirmation in the work of Odlyzko in the 80's, so the GUE law is nowadays called the Montgomery-Odlyzko law[6].

A further step along this physical approach to RH was taken by Berry, who noticed a similarity between the formula yielding the fluctuations of the number of the zeros around its average position, and a formula giving the fluctuations of the energy levels of a Hamiltonian obtained by the quantization of a classical chaotic system[?]. The quantum chaos approach can explain the deviations from the GUE law found numerically by Odlyzko. By this model suffer an overall sign problem and this problem lead Connes to propose an abstract approach to the RH based on discrete mathematical objects known as adeles[7]. Those approaches give two possible physical realizations of the Riemann zeros, either as point like spectra or as missing spectra in a continuum. The next step came in 1999 when Berry and Keating[8] on one hand and Connes[7] on the other, proposed that the classical Hamiltonian $H = xp$, where x and p are the position and momenta of a 1D particle, is closely related to the Riemann zeros. They choose different regularizations of this Hamiltonian and gave different realization of the Riemann zeros. And all these semiclassical results are heuristic and lack so far of a consistent quantum version. And there are further work along this line[9].

In this paper, we will go along with other approach, that is to consider the zeros as the eigenvalues of the Dirac operator on a suitable manifold. With the heat kernel expansion we can get the dimension of this manifold and find that the manifold is fractal and has fractal dimension about $1.1 - 1.2$, closed to the 1 dimension. And our result don't dependent on the models.

2. The Riemann Zeta Zeros

The Riemann Zeta function is defined in the half-plane $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p(\text{prime})} \frac{1}{1 - p^{-s}}. \quad (2.1)$$

and has a meromorphic continuation to \mathbb{C} . This function satisfy the functional equation

$$\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \xi(1 - s). \quad (2.2)$$

It is known that the complex zeros of $\zeta(s)$ lie in the "critical strip" $0 < Re(s) < 1$, and Riemann hypothesis states that in fact all these zeros lie on the "critical line" $Re(s) = 1/2$, that is the t_j on 1.1 are all real. The asymptotic density of the zeros is given by

$$d(t) = \frac{1}{2\pi} \log\left(\frac{t}{2\pi}\right) + O\left(\frac{1}{t^2}\right) \quad (2.3)$$

and therefore that the mean spacing between the zeros decreases logarithmically with increasing t . Define the scaled Riemann zeros so as to have unit mean spacing, that is

$$u_j = \frac{t_j}{2\pi} \log \frac{t_j}{2\pi}. \quad (2.4)$$

We will consider those numbers as the eigenvalues of the Dirac operator on some conjectured manifold, and compare to the results from the random matrix theory and quantum chaos theory.

3. The fractal dimension of the Riemann zeros

The reason why we consider the zeros as the eigenvalues of the Dirac operator is that, from the functional equation 2.2 we know that if $1/2 + it_j$ is the zero of the Riemann zeta function, $1/2 - it_j$ is also the zero of the function, so $\pm t_j$ both are eigenvalues, thus we choose the Dirac operator instead of the Laplacian operator.

We assume that this Dirac operator has the similar heat kernel expansion as the usual one, and this assumption can be verified by our results. From the heat kernel

expansion we can define the fractal dimension as follows[5]:

$$D_s = -\frac{d \ln(\text{Trace}(e^{-D^2/\Lambda^2}))}{d \ln \Lambda}. \quad (3.1)$$

Substitute the scaled Riemann zeros u_j [10] into 3.1, and plot the figure of the fractal dimension varying with the Λ . From the above figure we can see that although the

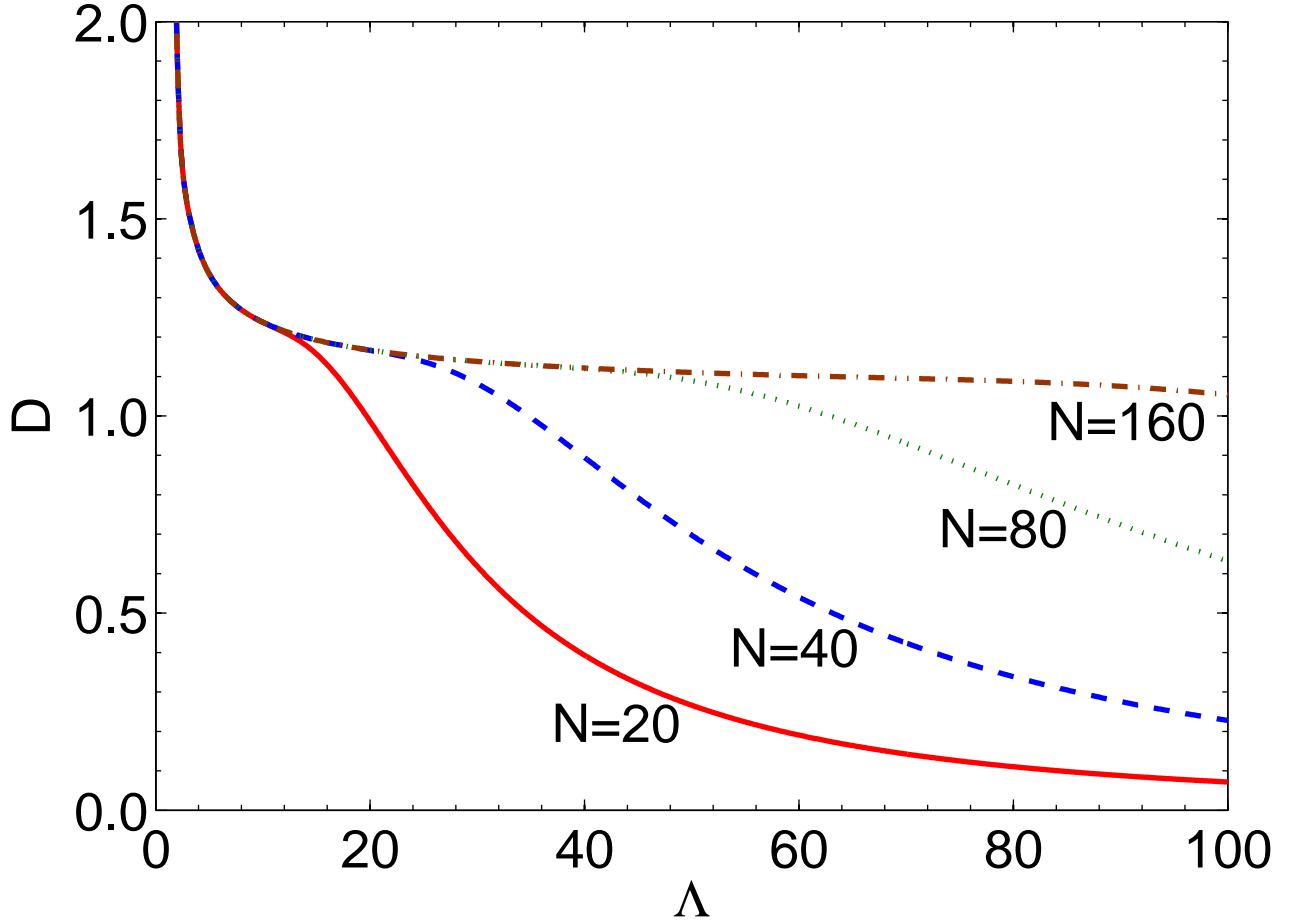


Figure 1: The fractal dimension of the Riemann zeta zeros for different numbers

fractal dimension general vary with the Λ , at the middle region, the curve is nearly flat, and correspondent to

$$D_s = 1.1 - 1.2 \quad (3.2)$$

In the previous paper[11], we show that for ordinary sphere, this region give the correct dimension, so here we also consider this number is the dimension of the

underling manifold. We also notice that for more zeros, the nearly flat region will be longer.

The GUE conjecture state that all statistics for zeta zeros and eigenvalues of Hermitian matrices match. And the numerical calculation performed by Odlyzko confirmed this conjecture. But it can't tell us exactly which Hermitian matrix gives the zeta zeros. On the other hand, the quantum chaos theory give some clues for the dynamics behind the zeta function, that is chaotic, time-asymmetry system. But the system is 1 dimension, and can't give the above figure. So we think that this approach is unsatisfying.

4. Conclusion

We get the fractal dimension from the scaled zeros of the Riemann zeta function, so it must be obeyed by any system which is used to model the Riemann zeta function. Unfortunately, the current physical models, such as the GUE matrices, quantum chaos system don't satisfy this picture.

The fractal dimension is about $D_s = 1.1 - 1.2$, not away from the integer 1, but show the fractal character of the Riemann zeta function. The similarity between the Riemann hypothesis and the quantum gravity has been point out by Connes and Marcolli in their book[12], and our result confirm this relations.

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